

# THE HIGHER ORDER ROGUE WAVE SOLUTIONS OF THE GERDJIKOV-IVANOV EQUATION

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**ABSTRACT.** We construct higher order rogue wave solutions for the Gerdjikov-Ivanov equation explicitly in term of determinant expression. Dynamics of both soliton and non-soliton solutions is discussed. A family of solutions with distinct structures are presented, which are new to the Gerdjikov-Ivanov equation.

**Key words:** Darboux transformation, higher order rogue wave, Gerdjikov-Ivanov equation.

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## 1. Introduction

Rogue wave is one type of natural disasters first found in deep ocean wave. Later, people observed similar phenomena in other physical breaches, such as optical physics, plasmas, capillary waves and so on [1–5]. There is a consensus that these rogue waves are the result of modulation instability waves. Meanwhile, breather solution usually comes from the instability of small amplitude perturbations that may grow in size to disastrous proportions. Therefore, in mathematical understanding, rogue wave can be treated as a limit case of Ma soliton when the space period tends to infinity, or of the Akhmediev breather as the time period approaches to infinity [6]. Due to both theoretical frame and reality application, it is imperative to do further study about rogue waves, especially, higher order rogue waves of different models.

The Peregrine soliton which is located in time-space plane is one of the formal ways to explain the surprise phenomenon mathematically [7]. The solution appears from a non-zero constant and disappears to the constant background as time approaching infinity, but it develops a localized hump with peak amplitude three times of average waves in the intermediate times. The object of higher order rogue wave is now under intense discussion by different approaches such as iterated Darboux transformation method [8,9], the algebra-geometric means [10], and generalized Darboux transformation [11]. Recently, the expression of the first order rogue wave solution and the figure of the second order rouge wave for the Gerdjikov-Ivanov (GI) equation was provided by the third and fifth authors of present paper [12]. However, the higher order rogue wave solutions for this equation have not been studied. The main aim of this paper is to discuss higher order rogue wave solutions and describe their different structures.

The nonlinear Schrödinger equation is one of the most important equations in physics, which can be derived from Ablowitz-Kaup-Newell-Segur system [15,16]. Considering the higher order nonlinear effects, the derivative nonlinear schrödignger equation with a polynomial spectral problem of arbitrary order [17] is regard as a model in a wide variety of fields such as weakly nonlinear dispersive water waves [18], nonlinear optics fibers [19–21], quantum field theory [22],

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plasmas [23]. The DNLS equations have three generic deformations, the DNLSI equation [24]:

$$iq_t - q_{xx} + i(q^2 q^*)_x = 0, \quad (1)$$

the DNLSII equation [25]:

$$iq_t + q_{xx} + iqq^* q_x = 0, \quad (2)$$

and the DNLSIII equation or the GI equation [26]:

$$iq_t + q_{xx} - iq^2 q_x^* + \frac{1}{2}q^3 q^{*2} = 0. \quad (3)$$

In many circumstances, Darboux transformation has been proved to be one powerful methods to obtain soliton solutions [13, 14], breather solutions and rational solutions. The Darboux transformation and its determinant expression for the GI equations have been given in [12, 27]. But there is not a straight extension construct higher order rogue wave solutions at the same eigenvalues. In this paper, we take a limit technique with respect to degenerate eigenvalues and Taylor expansion in Darboux transformation [11, 28, 29]. Based on this explicit method, we can further discuss the structure of solutions, both soliton and rogue wave.

This paper is organized as following: In section 2, we review the general  $n$ -fold Darboux transformation for the GI equation. In section 3, explicit solutions are constructed, such as soliton, breather, position solutions and higher order rogue wave with two parameters  $D_1$  and  $D_2$ . By choosing different values of  $D_1$  and  $D_2$ , we show four basic models, fundamental pattern, triangular structure, modified triangular structure and ring structure, and display their dynamical evolutions respectively in section 4. The conclusions and discussions are contained in the final section.

## 2. Darboux transformation for the Gerdjikov-Ivanov equation

In this section, we start with Lax pair of (3) to construct Darboux transformation. Considering the spectral problem

$$\begin{cases} \partial_x \psi = (J\lambda^2 + Q_1\lambda + Q_0)\psi = U\psi, \\ \partial_t \psi = (2J\lambda^4 + V_3\lambda^3 + V_2\lambda^2 + V_1\lambda + V_0)\psi = V\psi. \end{cases} \quad (4)$$

where

$$\psi = \begin{pmatrix} \phi \\ \varphi \end{pmatrix} = \begin{pmatrix} \phi(x, t, \lambda) \\ \varphi(x, t, \lambda) \end{pmatrix}, \quad J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} -\frac{1}{2}iqr & 0 \\ 0 & \frac{1}{2}iqr \end{pmatrix},$$

$$V_3 = 2Q_1, V_2 = Jqr, V_1 = \begin{pmatrix} 0 & iq_x \\ -ir_x & 0 \end{pmatrix}, V_0 = \begin{pmatrix} \frac{1}{2}(rq_x - qr_x) + \frac{1}{4}iq^2r^2 & 0 \\ 0 & -\frac{1}{2}(rq_x - qr_x) - \frac{1}{4}iq^2r^2 \end{pmatrix}.$$

here  $\lambda \in \mathbb{C}$ ,  $\psi$  is the eigenfunction of (4) corresponding to the eigenvalue  $\lambda$ . By the condition  $U_t - V_x + [U, V] = 0$ , we get

$$\begin{cases} iq_t + q_{xx} + iq^2 r_x + \frac{1}{2}q^3 r^2 = 0, \\ ir_t - r_{xx} + ir^2 q_x - \frac{1}{2}q^2 r^3 = 0. \end{cases} \quad (5)$$

This system admits the reduction  $r = -q^*$ , and (5) becomes just one equation which is the GI equation (3).

From gauge transformation, we can construct new solution from initial data, i.e, if there exist some non-singular  $T$ , such that

$$\begin{cases} U^{[1]} = (T_x + T U)T^{-1}. \\ V^{[1]} = (T_t + T V)T^{-1}. \end{cases} \quad (6)$$

where  $U^{[1]}$  and  $V^{[1]}$  have the same form as  $U$  and  $V$  with  $q$  and  $r$  replaced by certain  $q^{[1]}$  and  $r^{[1]}$ . Therefore, it is crucial to find an algebraic formula for  $T$  instead of the (6).

## 2.1 N-fold Darboux transformation for the GI system

The  $n$ -fold Darboux transformation for  $q^{[n]}, r^{[n]}$  of the GI system has been given in in [12]. Since we need this result for our paper, we cite the main theorem as follows.

**Theorem 1.** Let  $\Psi_i = \begin{pmatrix} \phi_i \\ \varphi_i \end{pmatrix}$  ( $i = 1, 2, \dots, n$ ) be distinct solutions related to  $\lambda_i$  of the spectral problem, then  $(q^{[n]}, r^{[n]})$  given by the following formulae are new solutions of the GI system.

$$q^{[n]} = q + 2i \frac{\Omega_{11}}{\Omega_{12}}, r^{[n]} = r - 2i \frac{\Omega_{21}}{\Omega_{22}}. \quad (7)$$

Here, (1) for  $n = 2k$ ,

$$\begin{aligned} \Omega_{11} &= \begin{vmatrix} \phi_1 & \lambda_1 \phi_1 & \dots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \phi_1 & -\lambda_1^n \phi_1 \\ \phi_2 & \lambda_2 \phi_2 & \dots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \phi_2 & -\lambda_2^n \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_n & \lambda_n \phi_n & \dots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \phi_n & -\lambda_n^n \phi_n \end{vmatrix}, \\ \Omega_{12} &= \begin{vmatrix} \phi_1 & \lambda_1 \phi_1 & \dots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \phi_1 & \lambda_1^{n-1} \phi_1 \\ \phi_2 & \lambda_2 \phi_2 & \dots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \phi_2 & \lambda_2^{n-1} \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_n & \lambda_n \phi_n & \dots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \phi_n & \lambda_n^{n-1} \phi_n \end{vmatrix}, \\ \Omega_{21} &= \begin{vmatrix} \varphi_1 & \lambda_1 \phi_1 & \dots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \phi_1 & -\lambda_1^n \phi_1 \\ \varphi_2 & \lambda_2 \phi_2 & \dots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \phi_2 & -\lambda_2^n \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_n & \lambda_n \phi_n & \dots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \phi_n & -\lambda_n^n \phi_n \end{vmatrix}, \\ \Omega_{22} &= \begin{vmatrix} \varphi_1 & \lambda_1 \phi_1 & \dots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \phi_1 & \lambda_1^{n-1} \phi_1 \\ \varphi_2 & \lambda_2 \phi_2 & \dots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \phi_2 & \lambda_2^{n-1} \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_n & \lambda_n \phi_n & \dots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \phi_n & \lambda_n^{n-1} \phi_n \end{vmatrix}; \end{aligned} \quad (8)$$

(2) for  $n = 2k + 1$ ,

$$\Omega_{11} = \begin{vmatrix} \varphi_1 & \lambda_1 \phi_1 & \dots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \phi_1 & -\lambda_1^n \phi_1 \\ \varphi_2 & \lambda_2 \phi_2 & \dots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \phi_2 & -\lambda_2^n \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_n & \lambda_n \phi_n & \dots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \phi_n & -\lambda_n^n \phi_n \end{vmatrix}, \quad (9)$$

$$\begin{aligned}
\Omega_{12} &= \begin{vmatrix} \varphi_1 & \lambda_1 \phi_1 & \dots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \varphi_1 & \lambda_1^{n-1} \varphi_1 \\ \varphi_2 & \lambda_2 \phi_2 & \dots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \varphi_2 & \lambda_2^{n-1} \varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_n & \lambda_n \phi_n & \dots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \varphi_n & \lambda_n^{n-1} \varphi_n \end{vmatrix}, \\
\Omega_{21} &= \begin{vmatrix} \phi_1 & \lambda_1 \varphi_1 & \dots & \lambda_1^{n-3} \varphi_1 & \lambda_1^{n-2} \phi_1 & -\lambda_1^n \varphi_1 \\ \phi_2 & \lambda_2 \varphi_2 & \dots & \lambda_2^{n-3} \varphi_2 & \lambda_2^{n-2} \phi_2 & -\lambda_2^n \varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_n & \lambda_n \varphi_n & \dots & \lambda_n^{n-3} \varphi_n & \lambda_n^{n-2} \phi_n & -\lambda_n^n \varphi_n \end{vmatrix}, \\
\Omega_{22} &= \begin{vmatrix} \phi_1 & \lambda_1 \varphi_1 & \dots & \lambda_1^{n-3} \varphi_1 & \lambda_1^{n-2} \phi_1 & \lambda_1^{n-1} \phi_1 \\ \phi_2 & \lambda_2 \varphi_2 & \dots & \lambda_2^{n-3} \varphi_2 & \lambda_2^{n-2} \phi_2 & \lambda_2^{n-1} \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_n & \lambda_n \varphi_n & \dots & \lambda_n^{n-3} \varphi_n & \lambda_n^{n-2} \phi_n & \lambda_n^{n-1} \phi_n \end{vmatrix}.
\end{aligned}$$

To solve the GI equation, we need to preserve the reduction condition  $r = -q^*$ , i.e., under  $n$  steps of Darboux transformation,  $r^{[n]} = -q^{[n]*}$ . Therefore, we can choose eigenfunctions  $\psi_k = \begin{pmatrix} \phi_k \\ \varphi_k \end{pmatrix}$  as follows [12]:

- (1)  $\lambda_k = -\lambda_k^*$ , and  $\psi_k = \begin{pmatrix} \phi_k \\ \varphi_k \end{pmatrix} = \begin{pmatrix} \varphi_k^* \\ \phi_k^* \end{pmatrix}$ .
- (2)  $\lambda_{2k} = -\lambda_{2k-1}^*$ , and  $\psi_{2k} = \begin{pmatrix} \phi_{2k} \\ \varphi_{2k} \end{pmatrix} = \begin{pmatrix} -\varphi_{2k-1}^* \\ \phi_{2k-1}^* \end{pmatrix}$ .

By this choice, we can construct soliton, breather, position solutions of the GI equation. But to get higher order rogue wave, we need to modify above Darboux transformation. At that time, the eigenfunctions and eigenvalues are no longer arbitrary, and only condition (2) works. We will discuss them in detail in the next section.

### 3. Solutions of the Gerdjikov-Ivanov equation

#### 3.1 Solution with vanishing boundary condition

We start with the soliton solutions by (7). Let  $q = 0$ , solutions of the spectral problem (4) with eigenvalues  $\lambda_k$  are solved as

$$\psi_k = \begin{pmatrix} \phi_k \\ \varphi_k \end{pmatrix}, \quad \phi_k = \exp(-i(\lambda_k^2 x + 2\lambda_k^4 t)), \quad \varphi_k = \exp(i(\lambda_k^2 x + 2\lambda_k^4 t)). \quad (10)$$

Example:

- $n = 1$ .  $\lambda_1^* = -\lambda_1, \lambda_1 = i\beta_1$  and  $\psi_1 = \begin{pmatrix} \phi_1 \\ \phi_1^* \end{pmatrix}$ , then

$$q^{[1]} = 2\beta_1 \exp^{-2i\beta_1^2(-x+2\beta_1^2 t)}.$$

This is a plane wave with constant amplitude.

- $n = 2$ .

(1) Let  $\lambda_2 = \lambda_1^*$ ,  $\lambda_1 = \alpha_1 + i\beta_1$  and  $\psi_1 = \begin{pmatrix} -\phi_1 \\ \varphi_1 \end{pmatrix}$ ,  $\psi_2 = \begin{pmatrix} -\varphi_1^* \\ \phi_1^* \end{pmatrix}$ , then

$$q^{[2]} = -8 \frac{e^{F_1} \alpha_1 \beta_1}{-e^{F_2} \alpha_1 + i e^{F_2} \beta_1 - e^{-F_2} \alpha_1 - i e^{-F_2} \beta_1} \quad (11)$$

with

$$\begin{aligned} F_1 &= -2i(\alpha_1^2 x + 2\alpha_1^4 t - 12\alpha_1^2 \beta_1^2 t - \beta_1^2 x + 2\beta_1^4 t), \\ F_2 &= 4\alpha_1 \beta_1 (4t\alpha_1^2 - 4t\beta_1^2 + x). \end{aligned} \quad (12)$$

It is a line soliton, and its trajectory is

$$x = -4t(\alpha_1 - \beta_1)(\alpha_1 + \beta_1),$$

on the  $(x, t)$  plane. Let  $\alpha_1 \rightarrow 0$  in (11), we obtain a rational travelling solution

$$q^{[2]} = 4 \frac{\beta_1 e^{-2i\beta_1^2(-x+2t\beta_1^2)}}{1 - 4i\beta_1^2 x + 16i\beta_1^4 t}, \quad (13)$$

with an arbitrary real constant  $\beta_1$ . Its trajectory is defined explicitly by

$$x = 4\beta_1^2 t$$

on the  $(x, t)$  plane. The above two solutions is plotted in Fig. 1(a) and Fig. 1(b).

(2) Let  $\lambda_1 = -\lambda_1^* = i\beta_1$ ,  $\lambda_2 = -\lambda_2^* = i\beta_2$ , then  $\psi_1 = \begin{pmatrix} \phi_1 \\ \phi_1^* \end{pmatrix}$ ,  $\psi_2 = \begin{pmatrix} \phi_2 \\ \phi_2^* \end{pmatrix}$ . we obtain a soliton solution

$$q^{[2]} = -2 \frac{e^{-i(-\beta_1^2 x + 2\beta_1^4 t - \beta_2^2 x + 2\beta_2^4 t)} (-\beta_2^2 + \beta_1^2)}{e^{-i(\beta_1 - \beta_2)(\beta_1 + \beta_2)(2t\beta_1^2 - x + 2\beta_2^2 t)} \beta_2 - \beta_1 e^{i(\beta_1 - \beta_2)(\beta_1 + \beta_2)(2t\beta_1^2 - x + 2\beta_2^2 t)}}. \quad (14)$$

Its trajectory is defined explicitly as

$$x = 2t(\beta_1^2 + \beta_2^2)$$

in the  $(x, t)$  plane. It is plotted in Fig. 1(c).

- $n = 3$ . Let  $\lambda_i = -\lambda_i^*$ ,  $i = 1, 2, 3$ , we get a one-soliton solution. While a  $\lambda_1 = -\lambda_1^*$ ,  $\lambda_3 = \lambda_2^*$ , we obtain a quasi-periodic solution. Since the analytic expressions are clumsy and tedious, we plot a particular quasi-periodic solution in Fig. 2(a).
- $n = 4$ . Under the different reduction conditions, we obtain three kinds of solutions. One of them is a soliton solution under the quasi-periodic background, the others are second soliton solutions. A soliton solution under the quasi-periodic background is plotted in Fig. 2(b).

**Remark.** If the eigenvalues share the same value,  $T$  becomes degenerate. Next, we will consider the degenerate cases by limit technique.

Given  $\psi = \psi(\lambda)$ , define  $\psi[i, j, k]$  by

$$\lambda^j \psi = \psi[i, j, 0] + \psi[i, j, 1]\epsilon + \psi[i, j, 2]\epsilon^2 + \cdots + \psi[i, j, k]\epsilon^k + \cdots, \quad (15)$$

with

$$\psi[i, j, k] = \frac{1}{k!} \frac{\partial^k}{\partial \epsilon^k} ((\lambda_i + \epsilon)^j \psi(\lambda_i + \epsilon)).$$

In other words,  $\psi[i, j, k]$  is the coefficient of  $\epsilon^k$  if we expand  $\lambda^j \psi$  at  $\lambda = \lambda_i$  by Taylor expansion.

Now, we consider the degenerate case of  $n = 4$ . According to the formula,

$$q^{[4]} = \frac{2i\delta_{11}}{\delta_{12}} \quad (16)$$

with

$$\delta_{11} = \begin{vmatrix} \phi_1 & \lambda_1 \psi_1 & \lambda_1^2 \phi_1 & -\lambda_1^4 \phi_1 \\ \phi_2 & \lambda_2 \psi_2 & \lambda_2^2 \phi_2 & -\lambda_2^4 \phi_2 \\ \phi_3 & \lambda_3 \psi_3 & \lambda_3^2 \phi_3 & -\lambda_3^4 \phi_3 \\ \phi_4 & \lambda_4 \psi_4 & \lambda_4^2 \phi_4 & -\lambda_4^4 \phi_4 \end{vmatrix},$$

$$\delta_{12} = \begin{vmatrix} \phi_1 & \lambda_1 \psi_1 & \lambda_1^2 \phi_1 & \lambda_1^3 \psi_1 \\ \phi_2 & \lambda_2 \psi_2 & \lambda_2^2 \phi_2 & \lambda_2^3 \psi_2 \\ \phi_3 & \lambda_3 \psi_3 & \lambda_3^2 \phi_3 & \lambda_3^3 \psi_3 \\ \phi_4 & \lambda_4 \psi_4 & \lambda_4^2 \phi_4 & \lambda_4^3 \psi_4 \end{vmatrix}.$$

Notice that these eigenfunction vectors  $\psi_i (i = 1, 3)$  have the same formula except only different eigenvalue  $\lambda_i$ . Thus, if we set  $\lambda_1 = \lambda_3$ , the determinants are linearly dependent. Then, we will obtain a trivial solution  $q^{[4]} = 0$ . To get non-trivial solution, we need to adopt the Taylor expansion technique.

The main process: **i)** For the first(second) row, we can substitute the eigenvalue  $\lambda_1 (\lambda_2 = \lambda_1^*)$  and eigenfunction  $\psi_1 (\psi_2)$  directly. **ii)**  $\lambda_3 \rightarrow \lambda_1 (\lambda_4 \rightarrow \lambda_2)$ , we expand the elements of the third(forth) row at  $\lambda_3 = \lambda_1 + \epsilon (\lambda_4 = \lambda_2 + \epsilon)$  by Taylor expansion at first, then subtract the first(second) row from the third(forth) row. **iii)** Taking  $\epsilon \rightarrow 0$ , the terms with higher order of  $\epsilon$  will vanish and the new expression of  $q^{[4]}$  is obtained as the same form as eq. (16), but with the values for  $\delta_{11}$  and  $\delta_{12}$  given by

$$\delta_{11} = \begin{vmatrix} \phi[1, 0, 0] & \varphi[1, 1, 0] & \phi[1, 2, 0] & -\phi[1, 4, 0] \\ \phi[2, 0, 0] & \varphi[2, 1, 0] & \phi[2, 2, 0] & -\phi[2, 4, 0] \\ \phi[1, 0, 1] & \varphi[1, 1, 1] & \phi[1, 2, 1] & -\phi[1, 4, 1] \\ \phi[2, 0, 1] & \varphi[2, 1, 1] & \phi[2, 2, 1] & -\phi[2, 4, 1] \end{vmatrix},$$

$$\delta_{12} = \begin{vmatrix} \phi[1, 0, 0] & \varphi[1, 1, 0] & \phi[1, 2, 0] & \varphi[1, 3, 0] \\ \phi[2, 0, 0] & \varphi[2, 1, 0] & \phi[2, 2, 0] & \varphi[2, 3, 0] \\ \phi[1, 0, 1] & \varphi[1, 1, 1] & \phi[1, 2, 1] & \varphi[1, 3, 1] \\ \phi[2, 0, 1] & \varphi[2, 1, 1] & \phi[2, 2, 1] & \varphi[2, 3, 1] \end{vmatrix}. \quad (17)$$

Substituting  $\lambda_1 = \alpha_1 + i\beta_1$  and eigenfunctions of eq. (10) into the above formula. we obtain a positon solution

$$q^{[4]} = \frac{G_1 + iG_2}{G_3 + iG_4}, \quad (18)$$

where

$$\begin{aligned}
G_1 &= \alpha_1 \beta_1, \\
G_2 &= -16 \cos(F_2) \alpha_1^3 (-32 \beta_1^3 \alpha_1^2 t \sinh(F_1) + 16 \beta_1 \alpha_1^4 t \sinh(F_1) + 4 \beta_1^3 x \sinh(F_1) + 4 \alpha_1^2 \beta_1 x \sinh(F_1) \\
&\quad - 48 \beta_1^5 t \sinh(F_1) - \alpha_1 \cosh(F_1)) \beta_1, \\
G_3 &= 16 \alpha_1^4 \beta_1^2 x + 64 \alpha_1^6 t \beta_1^2 + 2 \alpha_1 \beta_1^3 \sinh(2 F_2) - 384 \alpha_1^4 \beta_1^4 t - 16 \alpha_1^2 \beta_1^4 x \\
&\quad + 2 \alpha_1^3 \beta_1 \sinh(2 F_2) + 64 \alpha_1^2 \beta_1^6 t, \\
G_4 &= -256 \alpha_1^2 x \beta_1^8 t + 32 \alpha_1^2 x^2 \beta_1^6 + \alpha_1^4 + 64 \beta_1^4 x^2 \alpha_1^4 - 256 \alpha_1^4 x \beta_1^6 t + 256 \alpha_1^8 \beta_1^2 x t \\
&\quad + 3072 \beta_1^6 t^2 \alpha_1^6 + 2048 \beta_1^4 t^2 \alpha_1^8 + 32 \alpha_1^6 x^2 \beta_1^2 + 512 \alpha_1^{10} t^2 \beta_1^2 + 512 \beta_1^{10} t^2 \alpha_1^2 \\
&\quad + 2048 \alpha_1^4 t^2 \beta_1^8 + \beta_1^4 + 256 \beta_1^4 x \alpha_1^6 t + \alpha_1^4 \cosh(2 F_2) - \beta_1^4 \cosh(2 F_2), \\
F_1 &= 4 \alpha_1 \beta_1 (x + 4 t \alpha_1^2 - 4 t \beta_1^2), \\
F_2 &= -24 \alpha_1^2 t \beta_1^2 + 4 \beta_1^4 t + 4 \alpha_1^4 t - 2 \beta_1^2 x + 2 \alpha_1^2 x.
\end{aligned}$$

Its dynamical evolution is plotted in Fig. 3(a).

If we let  $\alpha_1 \rightarrow 0$  in above procedure, we will get the second order rational solution. With these parameters, the general solution can be given as following:

$$q^{[4]} = \frac{G_1}{G_2 + i G_3} \exp(-2 i \beta_1^2 (-x + 2 t \beta_1^2)), \quad (19)$$

with

$$\begin{aligned}
G_1 &= -8 \beta_1 (-12 i \beta_1^2 x + 768 i \beta_1^8 x^2 t + 4096 i \beta_1^{12} t^3 - 64 i \beta_1^6 x^3 - 48 i \beta_1^4 t - 3072 i \beta_1^{10} x t^2 - 3 + 2304 t^2 \beta_1^8 \\
&\quad - 768 x \beta_1^6 t + 48 \beta_1^4 x^2), \\
G_2 &= 3 - 768 x \beta_1^6 t + 4096 \beta_1^{10} t x^3 - 24576 \beta_1^{12} x^2 t^2 + 65536 \beta_1^{14} x t^3 + 4608 t^2 \beta_1^8 - 65536 \beta_1^{16} t^4 \\
&\quad - 256 \beta_1^8 x^4 + 96 \beta_1^4 x^2, \\
G_3 &= -12288 \beta_1^{10} x t^2 + 16384 \beta_1^{12} t^3 - 256 \beta_1^6 x^3 - 48 \beta_1^2 x + 3072 \beta_1^8 x^2 t + 576 \beta_1^4 t.
\end{aligned}$$

The equations (18) and (19) represent the interaction of solitons and rational solitons, respectively. A simplify analysis shows that they possess phase shift when  $t \rightarrow \pm \infty$ . Which is different from general 2-soliton solutions. They are shown in Fig. 3.

This process works for general  $n$ .

**Theorem 2.** For  $n = 2k$ ,  $\psi = \begin{pmatrix} \phi \\ \varphi \end{pmatrix}$  is eigenfunction vector of the GI system, the expression of  $n$ -positon solutions in terms of determinant is obtained:

$$q^{[n]} = 2i \frac{\delta_{11}}{\delta_{12}}, \quad (20)$$

where

$$\delta_{11} = \begin{vmatrix} \phi[1, 0, 0] & \varphi[1, 1, 0] & \cdots & \varphi[1, n-3, 0] & \phi[1, n-2, 0] & -\phi[1, n, 0] \\ \phi[2, 0, 0] & \varphi[2, 1, 0] & \cdots & \varphi[2, n-3, 0] & \phi[2, n-2, 0] & -\phi[2, n, 0] \\ \phi[1, 0, 1] & \varphi[1, 1, 1] & \cdots & \varphi[1, n-3, 1] & \phi[1, n-2, 1] & -\phi[1, n, 1] \\ \phi[2, 0, 1] & \varphi[2, 1, 1] & \cdots & \varphi[2, n-3, 1] & \phi[2, n-2, 1] & -\phi[2, n, 1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi[1, 0, k-1] & \varphi[1, 1, k-1] & \cdots & \varphi[1, n-3, k-1] & \phi[1, n-2, k-1] & -\phi[1, n, k-1] \\ \phi[2, 0, k-1] & \varphi[2, 1, k-1] & \cdots & \varphi[2, n-3, k-1] & \phi[2, n-2, k-1] & -\phi[2, n, k-1] \end{vmatrix},$$

$$\delta_{12} = \begin{vmatrix} \phi[1, 0, 0] & \varphi[1, 1, 0] & \cdots & \varphi[1, n-3, 0] & \phi[1, n-2, 0] & \varphi[1, n-1, 0] \\ \phi[2, 0, 0] & \varphi[2, 1, 0] & \cdots & \varphi[2, n-3, 0] & \phi[2, n-2, 0] & \varphi[2, n-1, 0] \\ \phi[1, 0, 1] & \varphi[1, 1, 1] & \cdots & \varphi[1, n-3, 1] & \phi[1, n-2, 1] & \varphi[1, n-1, 1] \\ \phi[2, 0, 1] & \varphi[2, 1, 1] & \cdots & \varphi[2, n-3, 1] & \phi[2, n-2, 1] & \varphi[2, n-1, 1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi[1, 0, k-1] & \varphi[1, 1, k-1] & \cdots & \varphi[1, n-3, k-1] & \phi[1, n-2, k-1] & \varphi[1, n-1, k-1] \\ \phi[2, 0, k-1] & \varphi[2, 1, k-1] & \cdots & \varphi[2, n-3, k-1] & \phi[2, n-2, k-1] & \varphi[2, n-1, k-1] \end{vmatrix}.$$

*Proof.* For the entries in the first column of  $\Omega_{11}(8)$ ,

$$\begin{aligned} \phi_1 &= \phi[1, 0, 0], \\ \phi_2 &= \phi[2, 0, 0], \\ \phi_3 &= \phi[1, 0, 0] + \phi[1, 0, 1]\epsilon, \\ \phi_4 &= \phi[2, 0, 0] + \phi[2, 0, 1]\epsilon, \\ &\vdots \\ \phi_{n-1} &= \phi[1, 0, 0] + \phi[1, 0, 1]\epsilon + \phi[1, 0, 2]\epsilon^2 + \cdots + \phi[1, 0, k-1]\epsilon^{k-1}, \\ \phi_n &= \phi[2, 0, 0] + \phi[2, 0, 1]\epsilon + \phi[2, 0, 2]\epsilon^2 + \cdots + \phi[2, 0, k-1]\epsilon^{k-1}, \end{aligned}$$

Taking the similar procedure to the other entries in  $\Omega_{11}, \Omega_{12}$ . Finally, the  $q^{[n]}$  can be obtained through simple calculation and certain limit.  $\square$

Let  $\alpha_1 \rightarrow 0$  in above formula, we will get the  $n$ -rational solution.

### 3.2 Solutions with non-vanishing boundary condition

In this section, we consider solutions from non-trivial seed, and this will give higher order rogue wave. In general, we start with  $q = ce^{i[ax - (a^2 + c^2 a - \frac{c^4}{2})t]}$ ,  $a, c \in \mathbb{C}^2$ . Then the corresponding eigenfunctions  $\psi_k$  associated with  $\lambda_k$

$$\begin{pmatrix} \phi_k(\lambda_k) \\ \varphi_k(\lambda_k) \end{pmatrix} = \begin{pmatrix} D_1 \varpi_1(\lambda_k)[1, k] + D_2 \varpi_2(\lambda_k)[1, k] - D_2 \varpi_1^*(\lambda_k^*)[2, k] - D_1 \varpi_2^*(\lambda_k^*)[2, k] \\ D_1 \varpi_1(\lambda_k)[2, k] + D_2 \varpi_2(\lambda_k)[2, k] + D_2 \varpi_1^*(\lambda_k^*)[1, k] + D_1 \varpi_2^*(\lambda_k^*)[1, k] \end{pmatrix}, \quad (21)$$

where



$$\begin{pmatrix} \varpi_1(\lambda_k)[1, k] \\ \varpi_1(\lambda_k)[2, k] \end{pmatrix} = \begin{pmatrix} \exp(-c_1(ta - x - 2t\lambda_k^2) + \frac{i(ax - (a^2 + c^2a - \frac{c^4}{2})t)}{2}) \\ \frac{(ia - ic^2 + 2i\lambda_k^2 + 2c_1)}{2\lambda_k c} \exp(-c_1(ta - x - 2t\lambda_k^2) - \frac{i(ax - (a^2 + c^2a - \frac{c^4}{2})t)}{2}) \end{pmatrix},$$

$$\begin{pmatrix} \varpi_2(\lambda_k)[1, k] \\ \varpi_2(\lambda_k)[2, k] \end{pmatrix} = \begin{pmatrix} \exp(c_1(ta - x - 2t\lambda_k^2) + \frac{i(ax - (a^2 + c^2a - \frac{c^4}{2})t)}{2}) \\ \frac{(ia - ic^2 + 2i\lambda_k^2 - 2c_1)}{2\lambda_k c} \exp(c_1(ta - x - 2t\lambda_k^2) - \frac{i(ax - (a^2 + c^2a - \frac{c^4}{2})t)}{2}) \end{pmatrix},$$

$$\varpi_1(\lambda_k) = \begin{pmatrix} \varpi_1(\lambda_k)[1, k] \\ \varpi_1(\lambda_k)[2, k] \end{pmatrix}, \quad \varpi_2(\lambda_k) = \begin{pmatrix} \varpi_2(\lambda_k)[1, k] \\ \varpi_2(\lambda_k)[2, k] \end{pmatrix},$$

$$c_1 = \frac{\sqrt{-c^4 + 2c^2a - a^2 - 4a\lambda_k^2 - 4\lambda_k^4}}{2}. \quad (22)$$

In the following, we set  $D_1 = 1$  and  $D_2 = 1$ . Under the circumstance, Xu *et. al.* [12] have obtained two kinds of breather solutions. One is time periodic breather, and the other is space periodic breather. Besides, the first order and the second order rogue wave solutions have also been obtained by a certain limit from breather solution. However, the method is difficult to calculate the higher order rogue waves. What we do here is to take the limit in the determinant expression of solution for the GI equation directly, which is the same method as we do for the NLS equation [29].

**Proposition 1.** Let  $n = 2k$ ,  $\psi = \begin{pmatrix} \phi \\ \varphi \end{pmatrix}$  is eigenfunction vector, assuming

$$\lambda_1 = \frac{1}{2}\sqrt{c^2 - 2a} - \frac{1}{2}ic$$

and

$$\lambda_2 = \frac{1}{2}\sqrt{c^2 - 2a} + \frac{1}{2}ic,$$

the the formula of the  $n$ -th order rogue solution is obtained

$$q^{[n]} = q + 2i\frac{\delta_{11}}{\delta_{12}}, \quad (23)$$

where

$$\delta_{11} = \begin{vmatrix} \phi[1, 0, 1] & \varphi[1, 1, 1] & \cdots & \varphi[1, n-3, 1] & \phi[1, n-2, 1] & -\phi[1, n, 1] \\ \phi[2, 0, 1] & \varphi[2, 1, 1] & \cdots & \varphi[2, n-3, 1] & \phi[2, n-2, 1] & -\phi[2, n, 1] \\ \phi[1, 0, 2] & \varphi[1, 1, 2] & \cdots & \varphi[1, n-3, 2] & \phi[1, n-2, 2] & -\phi[1, n, 2] \\ \phi[2, 0, 2] & \varphi[2, 1, 2] & \cdots & \varphi[2, n-3, 2] & \phi[2, n-2, 2] & -\phi[2, n, 2] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi[1, 0, k] & \varphi[1, 1, k] & \cdots & \varphi[1, n-3, k] & \phi[1, n-2, k] & -\phi[1, n, k] \\ \phi[2, 0, k] & \varphi[2, 1, k] & \cdots & \varphi[2, n-3, k] & \phi[2, n-2, k] & -\phi[2, n, k] \end{vmatrix},$$

$$\delta_{12} = \begin{vmatrix} \phi[1, 0, 1] & \varphi[1, 1, 1] & \cdots & \varphi[1, n-3, 1] & \phi[1, n-2, 1] & \varphi[1, n-1, 1] \\ \phi[2, 0, 1] & \varphi[2, 1, 1] & \cdots & \varphi[2, n-3, 1] & \phi[2, n-2, 1] & \varphi[2, n-1, 1] \\ \phi[1, 0, 2] & \varphi[1, 1, 2] & \cdots & \varphi[1, n-3, 2] & \phi[1, n-2, 2] & \varphi[1, n-1, 2] \\ \phi[2, 0, 2] & \varphi[2, 1, 2] & \cdots & \varphi[2, n-3, 2] & \phi[2, n-2, 2] & \varphi[2, n-1, 2] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi[1, 0, k] & \varphi[1, 1, k] & \cdots & \varphi[1, n-3, k] & \phi[1, n-2, k] & \varphi[1, n-1, k] \\ \phi[2, 0, k] & \varphi[2, 1, k] & \cdots & \varphi[2, n-3, k] & \phi[2, n-2, k] & \varphi[2, n-1, k] \end{vmatrix}.$$

*Proof.* For the entries in the first column of  $\Omega_{11}(8)$ ,

$$\begin{aligned} \phi_1 &= \phi[1, 0, 1]\epsilon, \\ \phi_2 &= \phi[2, 0, 1]\epsilon, \\ \phi_3 &= \phi[1, 0, 1]\epsilon + \phi[1, 0, 2]\epsilon^2, \\ \phi_4 &= \phi[2, 0, 1]\epsilon + \phi[2, 0, 2]\epsilon^2, \\ &\vdots \\ \phi_{n-1} &= \phi[1, 0, 1]\epsilon + \phi[1, 0, 2]\epsilon^2 + \cdots + \phi[1, 0, k]\epsilon^k, \\ \phi_n &= \phi[2, 0, 1]\epsilon + \phi[2, 0, 2]\epsilon^2 + \cdots + \phi[2, 0, k]\epsilon^k, \end{aligned}$$

Taking the similar procedure to the other entries in  $\Omega_{11}, \Omega_{12}$ . Finally, the  $q^{[n]}$  can be obtained through simple calculation and certain limit.  $\square$

As application, we give some explicit solutions next.

- first order rogue wave

For  $n = 2$ , we get the first order rogue wave.

$$q^{[2]} = \frac{G_1}{G_2} \exp \left( \frac{1}{2} i (2ax - 2ta^2 - 2tc^2a + tc^4) \right), \quad (24)$$

with

$$\begin{aligned} G_1 &= -8c^2a^3t^2 + 12a^2c^4t^2 + 8c^2a^2tx - 8c^4tax - 2c^2ax^2 + 12iac^2t - 6ac^6t^2 - 3 \\ &\quad - 2ic^2x + 2c^4x^2 - 6ic^4t + 2c^8t^2, \\ G_2 &= -8c^2a^3t^2 + 12a^2c^4t^2 + 8c^2a^2tx - 8c^4tax - 2c^2ax^2 + 4ic^2ta - 6ac^6t^2 + 1 \\ &\quad - 2ic^2x + 2c^4x^2 + 2ic^4t + 2c^8t^2. \end{aligned}$$

Use this method, we also get similar result of [12]. Moreover, our method is general to product higher order rogue wave. Assuming  $x \rightarrow \infty$ ,  $t \rightarrow \infty$ , then  $q^{[2]} \rightarrow c^2$ . The maximum amplitude of  $q^{[2]}$  occurs at origin and is equal to  $9c^2$ . A first order rogue wave with particular parameters is shown in Fig. 4.

- Higher order Rogue Wave

Generally, the expression of rogue wave becomes more complicated with increasing  $n$ . For convenience, we use numerical simulations to discuss the higher order rogue wave. In the following, set  $a = 0$  and  $c = 1$ .

When  $n = 4$ , we can obtain the second order rogue wave according to the formula (23).

$$q^{[4]} = \frac{G_1}{G_2} \exp\left(\frac{1}{2}it\right), \quad (25)$$

with

$$\begin{aligned} G_1 = & 45 + 8t^6 - 24it^4x - 144it^3x^2 - 48it^2x^3 - 72itx^4 + 576it^2x + 288itx^2 \\ & - 144t^3x - 504t^2x^2 - 144tx^3 - 24ix^5 - 528it^3 + 414it + 90ix - 72it^5 \\ & + 48ix^3 + 24t^2x^4 + 24t^4x^2 + 504tx - 198x^2 - 486t^2 - 60x^4 - 60t^4 + 8x^6, \\ G_2 = & 9 + 48t^3x - 216t^2x^2 + 48tx^3 + 24it^5 - 24ix^5 + 24itx^4 + 48it^3x^2 + 198it \\ & - 24it^4x + 180t^4 + 24t^2x^4 + 24t^4x^2 + 8t^6 + 8x^6 - 12x^4 + 666t^2 + 90x^2 \\ & - 72tx + 336it^3 - 48ix^3 - 54ix + 288it^2x - 48it^2x^3. \end{aligned}$$

Besides, we can also obtain the  $k$ -th ( $k = 3, 4, 5, 6, 7$ ) order rogue wave solution of the GI equation. Since their analysis expressions are too cumbersome, we omit them. Their dynamical evolutions are displayed in Fig. 5. From the figures, we find these local peaks all own a high amplitude on their center, and there are many small peaks locating around the central peak. Compared with the higher order rogue wave of the NLS equation, the central high peak of higher order rogue wave of the GI equation looks like higher than the same order of the NLS equation. However, through detailed analysis, we find the amplitude of the  $k$ -th order rogue wave solution of the GI equation and the NLS equation are both  $2(k+1)c$  ( $c$  is the boundary condition of the seed solution) [29].

#### 4. The dynamics of Rogue Wave

In above section, we obtain the fundamental pattern of the higher order rogue wave solution for the GI equation with  $D_1 = 1$  and  $D_2 = 1$  in (21). Actually,  $D_1$  and  $D_2$  are arbitrary constant (or go to constant). In this section, we set  $D_1$  and  $D_2$  as following:

$$\begin{cases} D_1 = \exp(-ic_1(S_0 + S_1\epsilon + S_2\epsilon^2 + S_3\epsilon^3 + \cdots + S_{k-1}\epsilon^{k-1})), \\ D_2 = \exp(ic_1(S_0 + S_1\epsilon + S_2\epsilon^2 + S_3\epsilon^3 + \cdots + S_{k-1}\epsilon^{k-1})). \end{cases} \quad (26)$$

Here  $S_1, S_2, \dots, S_{k-1} \in \mathbb{C}$ . Although the terms with nonzero orders of  $\epsilon$  vanish in the  $\epsilon \rightarrow 0$  limit, analysis and numerics prove that their coefficients  $S_i$  ( $i = 1, 2, \dots, k-1$ ) have a crucial effect on the dynamics of higher order rogue wave. In the following, our main task is to talk about how do these parameters control these different spatial-temporal structures at the same order  $k$ .

##### 4.1. Solutions with one parameter

- $S_0$ : Fundamental pattern

When  $S_i = 0$  ( $i \neq 0$ ), we obtain a trivial translation. A special case for the first order rogue wave is shown in Fig. 6. It just change the location of the rogue wave. Actually we can shift the location of rogue wave to arbitrary position by hanging the value of  $S_0$ . The case for the NLS equation had been given in [30]. Moreover, the  $k$  order rogue waves have  $n(n+1)/2$  waves. Starting from large negative  $t$ ,  $n$  small peaks, then a row of  $n-1$  larger peaks etc., the central high amplitude wave appears. The process is

reversed in positive  $t$ . The dynamical evolution can be observed for the NLS equation distinctly in [31].

- $S_1$ : Two triangular structure

- (1) Triangular structure

Let all each coefficient  $S_i = 0$  except  $S_1$ . The higher order rogue wave solution of the GI equation split into a triangular structure. This structure of  $k$ -th order rogue wave contains  $k(k+1)/2$  first order fundamental patterns, which make up  $k$  successive arrays, and these arrays possess  $k, k-1, \dots, 1$  peaks respectively. These structures of the  $k$ -th ( $k = 2, 5, 7$ ) order rogue wave are shown in Fig. 7. From the Fig. 7(c), we observe seven arrays, and each of them has 7, 6, 5, 4, 3, 2, 1 local maxima respectively. And the orientation of these triangular structures in the  $(x, t)$ -plane remain the same. Our result is a general case of [32].

- (2) Modified triangular structure

Actually, the outer triangular is independent of the inner triangular in above triangular structure. For instance, when  $k = 5$ , the outer 12 local maxima and the inner 3 fundamental patterns both make up a triangular, but these two triangles are irrelevant. If we alter the appearance of (21) and mixing coefficients  $D_1$  and  $D_2$  as following:

$$\begin{pmatrix} \phi_k(\lambda_k) \\ \varphi_k(\lambda_k) \end{pmatrix} = \begin{pmatrix} D_1 \varpi_1(\lambda_k)[1, k] + D_1 \varpi_2(\lambda_k)[1, k] - D_2 \varpi_1^*(\lambda_k^*)[2, k] - D_2 \varpi_2^*(\lambda_k^*)[2, k] \\ D_1 \varpi_1(\lambda_k)[2, k] + D_1 \varpi_2(\lambda_k)[2, k] + D_2 \varpi_1^*(\lambda_k^*)[1, k] + D_2 \varpi_2^*(\lambda_k^*)[1, k] \end{pmatrix}, \quad (27)$$

with

$$\begin{cases} D_1 = \exp(-ic_1^2(S_0 + S_1\epsilon + S_2\epsilon^2 + S_3\epsilon^3 + \dots + S_{k-1}\epsilon^{k-1})), \\ D_2 = \exp(ic_1^2(S_0 + S_1\epsilon + S_2\epsilon^2 + S_3\epsilon^3 + \dots + S_{k-1}\epsilon^{k-1})). \end{cases} \quad (28)$$

Remark: the expression of (27) is different with (21).

The inner three peaks form a second fundamental pattern inversely. It is a new structure which has never been obtained for the NLS equation. Its dynamical evolution is shown in Fig. 8. For  $k$ -th order rogue wave, we conjecture that there are  $3k-3$  first order rogue wave locating the outer triangular shell and an  $(k-3)$ -th order rogue wave locating in the center.

- $S_{k-1}$ : Ring structure

If  $S_i = 0$  ( $i \neq k-1$ ), we obtain a ring structure. A high maximum peak is surrounded with some local maxima. By simple analysis, we find the peaks locating on the outer shell are all first order rogue waves, and the number of them is  $2k-1$ . Besides, the inner peak is a higher order rogue wave (except  $k = 3$ ), whose order is  $k-2$  (when  $k = 3$ , the inner peak is a first order rogue wave). For example, when  $k = 7$ , there are 13 first order rogue wave locating on the outer shell, and a fifth order rogue waves locates in the center. Some of these structures are shown in Fig. 9.

- (1)  $S_{k-1}, S_1$ : Ring-triangle

If  $S_1$  is also non-zero, the inner higher order rogue wave will become a triangular structure. For instance, when  $k = 5$ , the inner second order rogue wave is split into a triangle with  $S_4 = 5 \times 10^9$ ,  $S_1 = 500$ . Besides, when  $k = 6, 7$ , the similar structure is obtained. Therefore, we can conclude reasonably that the inner

higher order rogue wave is able to split into triangular structure when  $S_{k-1}$  is big enough and  $S_1 \neq 0$ . We display some of these special models in Fig. 10. Notably, the orientation of the triangle in 7-th ring-triangle model is different with the others.

(2)  $S_{k-1}, S_{k-3}$ : Multi-ring

Being similar to the above case, the inner higher order rogue wave can be split into a ring structure too. Keeping  $S_{k-1} \gg 0$  and setting  $S_{k-3} \neq 0$  ( $k > 3$ ), we will obtain a ring structure. In this case, if the central peak is still a higher order rogue wave, we can continue splitting it into first order fundamental models. For example, when  $k = 6$  with  $S_5 = 8 \times 10^9$ ,  $S_3 = 80000$ , we obtain a multi-ring structure with a second order rogue wave locating in the center. Its dynamical evolution is shown in Fig. 11. Under this circumstance, assuming  $S_1 \neq 0$ , then the second order rogue wave is split into a triangular structure. Its evolution is displayed in Fig. 12. Naturally, these similar structures are also able to found for  $k = 7$ . Since the third order rogue wave possesses two structures i.e. triangular structure and ring structure, the inner structure of the seventh can be both triangle structure and ring model. Their evolution is shown in Fig. 13.

In general, there are  $k - 1$  free parameters for  $k$ -th order rogue wave solution, which possesses  $2^k$  combinations. But there are only 4 basic models, and by choosing proper parameters, we also get dynamical evolutions of the 4 basic patterns. Therefore, we can get a hierarchy of higher order rogue wave of the GI equation with combination structure of above four basic models.

## 5. Conclusion

In this paper, we modified the generalized Darboux transformation to get explicit solutions for the GI equation. In the case of vanishing boundary condition, we obtain soliton, rational traveling soliton, breather type soliton, and soliton colliding with breather type soliton. The last two kinds of solutions are new for the GI equation. Moreover, we get the formula by applying the Taylor expansion and limit technique, when eigenvalues share the same value. Under the condition of non-vanishing boundary condition, we give the formula of the  $N$ -rogue wave solution for the GI equation in proposition 1. As applications, we give the expressions of higher order rogue wave solutions and discuss their structures. Further more, we generate solutions with different structures by adjusting the free parameters  $D_1$  and  $D_2$ . In summary, there are four basic patterns for higher order rogue waves of the GI equation: fundamental pattern, triangular structure, modified structure and ring structure. By choosing proper parameter, we can get solutions with combination structure of basic ones.

Our results clearly show the connection between shift parameters and structures of rogue wave due to the explicit formula. These types of solutions are new to the GI equation. Moreover, some basic models such as the fundamental pattern, ring structure and triangular structure have been found in higher rogue waves for other equations such as the NLS equation [9], but the modified triangular structure is new to the GI equation. Meanwhile, we get a family of solutions with combination structures. All of these will help us in founding univereal properties for rogue waves and better understanding of “waves appear from nowhere and disappear without a trace” [6].

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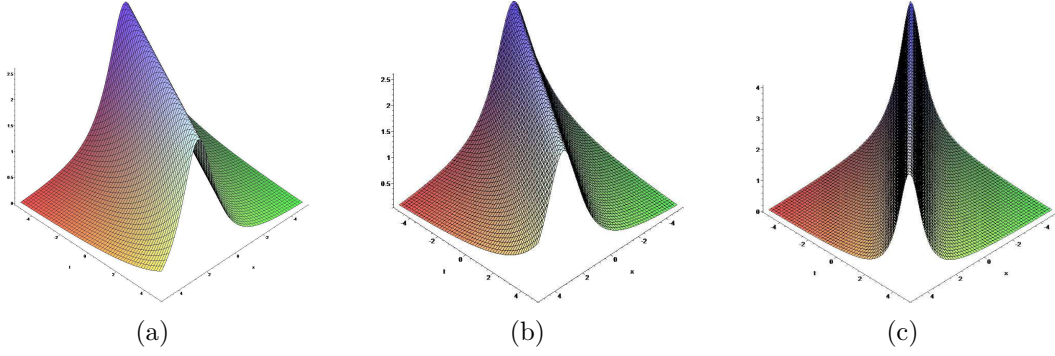


FIGURE 1. (Color online) The dynamics of  $|q^{[2]}|^2$ . (a) Eq. (11) soliton solution with  $\alpha_1 = 0.2$ ,  $\beta_1 = 0.4$ . (b) Eq. (13) rational travelling solution with  $\beta_1 = 0.4$ . (c) Eq. (14) soliton solution with  $\beta_1 = 0.45$ ,  $\beta_2 = 0.55$ .

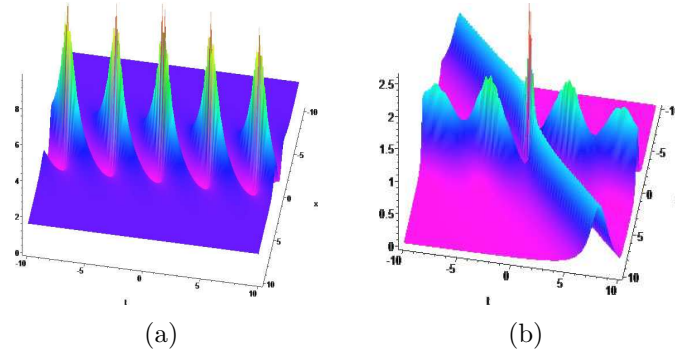


FIGURE 2. (Color online) (a) The dynamics of soliton solution quasi-periodic solution on the  $(x, t)$  plane. (b) soliton solution under the quasi-periodic background on the  $(x, t)$  plane.

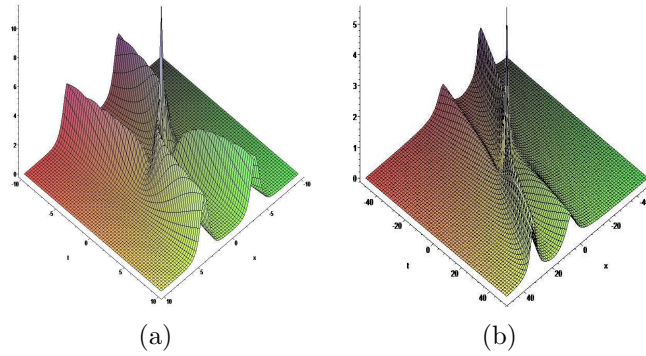


FIGURE 3. (Color online) The dynamics of 2-positon solution and 2-rational solution on the  $(x, t)$  plane. (a) Eq. (18) with  $\alpha_1 = 0.5$ ,  $\beta_1 = 0.5$ . (b) Eq. (18) with  $\alpha_1 \rightarrow 0$ .

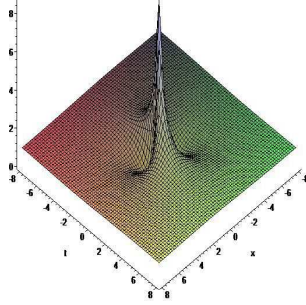


FIGURE 4. (Color online) The dynamics of the first order rogue wave solution  $|q^{[2]}|^2$  Eq. (24) on the  $(x, t)$  plane with  $a = 0$  and  $c = 1$ . The maximum amplitude of  $|q^{[2]}|^2$  occurs at  $t = 0$  and  $x = 0$  and are equal to 9.

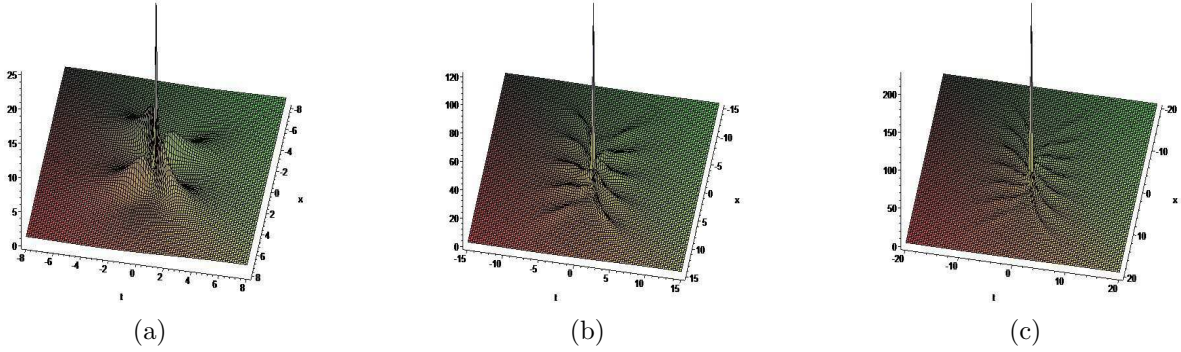


FIGURE 5. (Color online) The dynamics of higher order rogue wave. (a) The second order rogue wave. (b) The fifth order rogue wave. (f) The seventh order rogue wave.

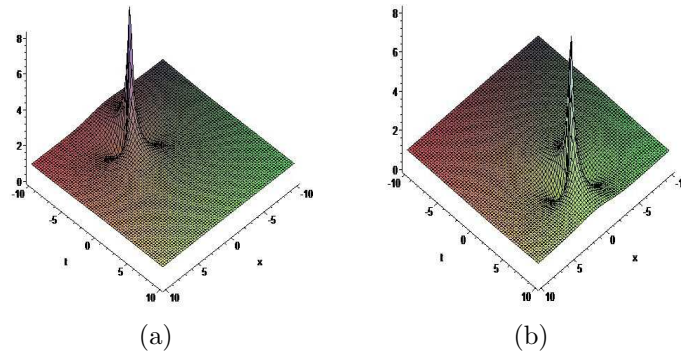


FIGURE 6. (Color online) The fundamental pattern of first order rogue wave. (a) The first order rogue wave with  $S_0 = 5$ , the maximum amplitude occurs at  $t = -5$  and  $x = 0$ . (b) The first order rogue wave with  $S_0 = -5$ , the maximum amplitude occurs at  $t = 5$  and  $x = 0$ .



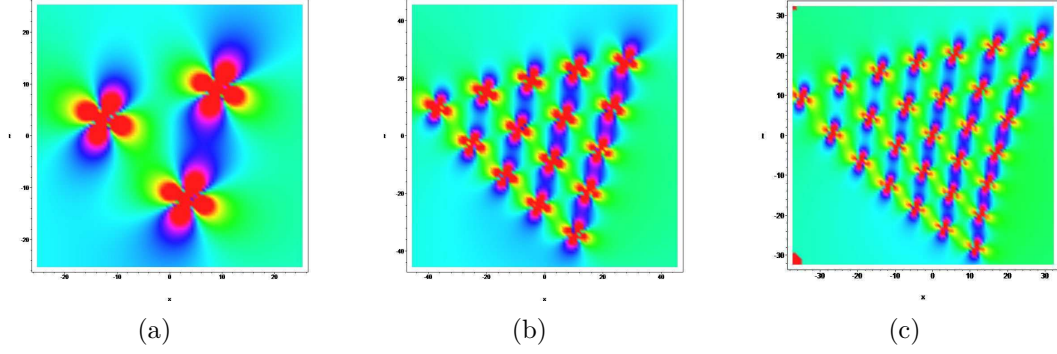


FIGURE 7. (Color online) The triangle structure of higher order rogue wave. (a) The second order rogue wave with  $S_1 = 1000$ . (b) The fifth order rogue wave with  $S_1 = 800$ . (c) The seventh order rogue wave with  $S_1 = 250$ .

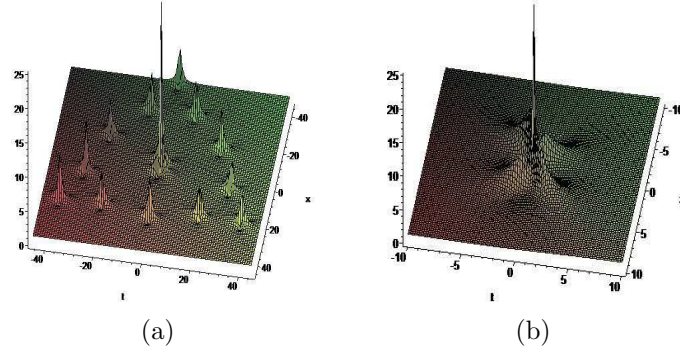


FIGURE 8. (Color online) The modified triangular structure with a higher order rogue wave locating in the center. (a) An overall profile with  $S_1 = 1000$ . (b) A local central profile of the right panel.

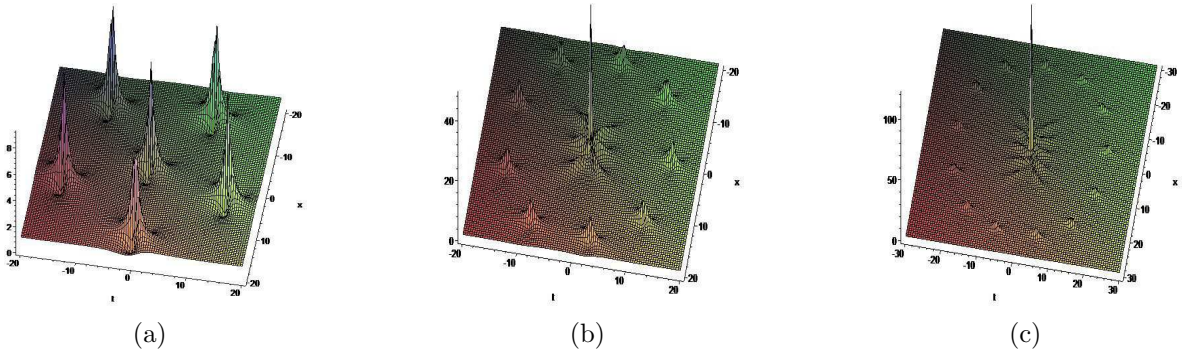


FIGURE 9. (Color online) The ring structure of higher order rogue wave. (a) The third order rogue wave with  $S_2 = 50000$ . (b) The fifth order rogue wave with  $S_4 = 5000000$ . (c) The seventh order rogue wave with  $S_6 = 5 \times 10^7$ .

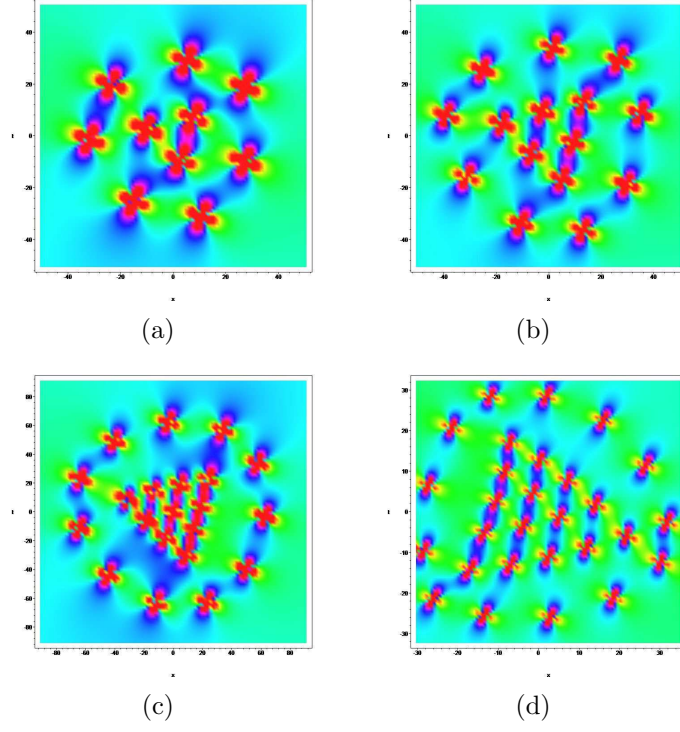


FIGURE 10. (Color online) The ring-triangle structure of higher order rogue wave. (a) The fourth order rogue wave with  $S_3 = 5 \times 10^7$ ,  $S_1 = 500$ . (b) The fifth order rogue wave with  $S_4 = 5 \times 10^9$ ,  $S_1 = 500$ . (c) The sixth order rogue wave with  $S_6 = 5 \times 10^{13}$ ,  $S_1 = 1000$ . (d) The seventh order rogue wave with  $S_7 = 1 \times 10^{11}$ ,  $S_1 = 150$ .

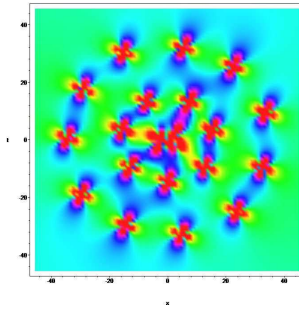


FIGURE 11. (Color online) The multi-ring model of the sixth order rogue wave with  $S_5 = 8 \times 10^9$ ,  $S_3 = 80000$ , the inner peak is still a higher order wave

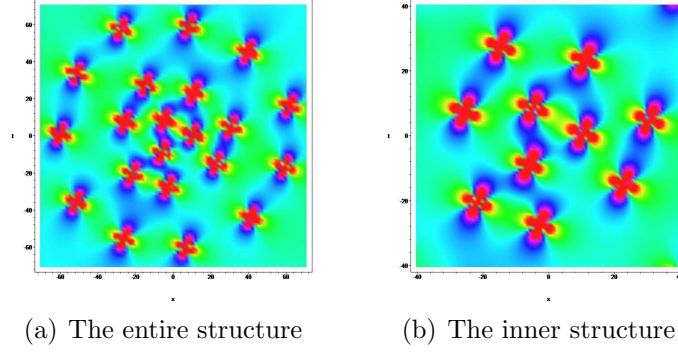


FIGURE 12. (Color online) The multi-ring model of the sixth order rogue wave, the inner higher rogue wave is split into a triangle.

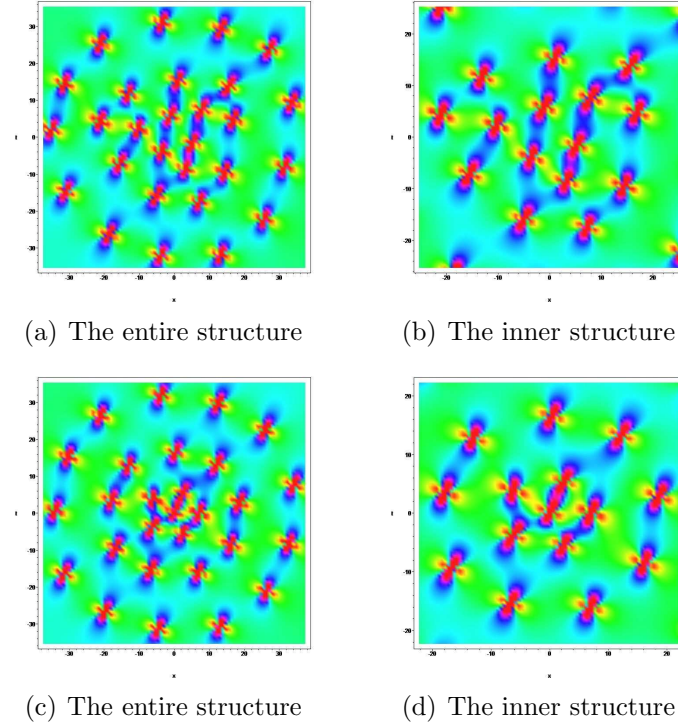


FIGURE 13. (Color online) The two kinds of multi-ring structure of 7-order rogue wave solution with three parameters. (a) The 7-order rogue wave with  $S_6 = 5 \times 10^{11}$ ,  $S_4 = 1 \times 10^7$ ,  $S_1 = 100$ . (b) The 7-th order rogue wave with  $S_6 = 5 \times 10^{11}$ ,  $S_4 = 1 \times 10^7$ ,  $S_2 = 1000$ .